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MORSE THEORY FOR FLOWS IN PRESENCE OF A SYMMETRY GROUP

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July, 1984

(Received April 25, 1984)

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MORSE THEORY FOR FLOWS IN PRESENCE OF A SYMMETRY GROUP

Filomena Pacella

Technical Summary Report #2717
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ABSTRACT

This paper contains results reported in a series of seminars given by the author at the University of Wisconsin-Madison. These concern Morse theory in the presence of symmetry. Different ways of studying an equivariant flow are investigated and, in particular, the equivariant Morse theory for flows is described.

This theory requires results on the cohomology of classifying spaces for finite groups which are also described here.

AMS (MOS) Subject Classifications: 20J06, 55N25, 57R70, 57S10, 58F25

Key Words: Morse Theory, Group actions, Equivariant flows, Equivariant cohomology

Work Unit Number 1 (Applied Analysis)

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

SIGNIFICANCE AND EXPLANATION

Morse theory is an important tool for studying dynamical systems. It often happens that the system under study (e.g. in celestial mechanics or quantum mechanics) is subject to some symmetries.

In this paper Morse theory for flows in the presence of a symmetry group is studied. In particular the so called "equivariant theory" is described. Then, using homological algebra, a method of treating finite groups is described.

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The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

MORSE THEORY FOR FLOWS IN PRESENCE OF A SYMMETRY GROUP

Filomena Pacella

0. Introduction.

This paper presents a discussion of Morse theory in the presence of a symmetry group given at the University of Wisconsin-Madison.

In these notes ideas and results from many different sources are collected and their application to the study of flows invariant under some group action is illustrated by some simple examples. Deeper applications require the extension of the theory of isolated invariant set to the equivariant case; this extension, which presents no serious difficulties, is also indicated here.

An interesting point, illustrated in the examples is that in the case of flows with symmetry there are (at least) three different ways to obtain "Morse relations" and that these generally give different information.

The subject is divided in six sections:

The first two are introductory. In section 1 I recall the main definitions about group actions on topological spaces and I give some easy examples. In section 2 Morse theory for flows is briefly sketched as exposed in [6] and [7] making the comparison with the classical Morse theory for gradient flows.

In section 3 the notion of equivariant flow is introduced and the different ways of studying it are presented. This section ends with the definition of nondegenerate critical manifold, as given in [1] and [3].

Section 4 treats the equivariant Morse theory for flows. In the exposition of this theory I have followed the ideas of [1], applying them to the case of an equivariant flow on a topological space invariant under the action of a group. In the second part of this

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section with very simple examples I illustrate the difference between the various ways of studying an equivariant flow. Section 5 and 6 deal with the cohomology of the classifying space of a finite group.

In section 6 it is shown that this cohomology is isomorphic to the cohomology of the group itself.

This motivates section 5 where the cohomology of a group is described.

The end of section 6 also contains the explicit computation of $H^*(BG)$ for finite abelian groups.

I would like to thank C. Conley for his encouragement in writing these notes.

1. Group actions on topological spaces.

Let X be a topological space and G a group with the multiplicative notation.

We will denote by $Aut(X)$ the group, under composition, of homeomorphisms from X to itself.

DEFINITION 1.1 - An action of G on X is an homomorphism:

$$\phi : G \to Aut(X) .$$

The homeomorphism corresponding to an element $g \in G$, is usually, denoted by:

$$\phi(g)(x) = g(x) \qquad x \in X .$$

When G is a topological group there is another way of defining an action on X which also considers the topology on G. Besides it distinguishes between left and right actions.

DEFINITION 1.2 - A left action of G on X is a map:

$$\mu : G \times X \to X, \qquad \mu(g,x) = gx$$

satisfying the following properties:

i) $1x = x, \quad 1 \in G, \quad x \in X$

ii) $g_1(g_2x) = (g_1g_2)x \quad g_1, g_2 \in G, \quad x \in X$

We speak about a right action if $\mu(g,x) = xg$ and i) and ii) are replaced by:

i)' $x1 = x, \quad x \in X, 1 \in G$

ii)' $(xg_1)g_2 = x(g_1g_2), \quad g_1, g_2 \in G \quad x \in X$

The difference between left and right actions is not just a matter of notation, since properties ii) and ii)' give a different order in applying g_1 and g_2. Hence if the group is not commutative, a left action is not generally a right action.

Given $x \in X$, we denote by $0(x)$ the "orbit" of x; that is, the set of those points in X which can be obtained from x using the action of the group:

$$0(x) = \{gx, \ g \in G\}.$$

Then, the quotient space X/G represents the set of all orbits.

The set $G_x = \{ g \in G, gx = x \}$ will be called the isotropy group of x; it is the set of elements in G which leave x fixed.

If G is a compact topological group, then G_x is a closed subgroup of G.

DEFINITION 1.3 - <u>The action of G on X is said to be free if:</u>

$g \in G$ <u>and</u> $g \neq 1 \Longrightarrow gx \neq x$ <u>for every</u> $x \in X$; <u>that is, if</u> $G_x = 1$ <u>for all</u> x.

If $x \in X$ and $p : G \to O(x)$ is the map given by $p(g) = gx$, then p is surjective. If the action is free, p is also injective. This implies that, when the action is free, every orbit looks like G.

DEFINITION 1.4 - <u>The action of G on X is said to be effective if:</u>

$$\bigcap_{x \in X} G_x = 1.$$

We also define the trivial action of G as the one which leaves everything fixed, that is: $\forall x$, $G_x = G$.

If G is a compact Lie group acting freely on a manifold X, then X/G is a manifold. However, if the action is not free, or the group is not compact this need not be the case.

In the case of groups with the discrete topology, we give the following definitions.

DEFINITION 1.4 - <u>An open set</u> $U \subset X$ <u>is called proper (under the action of G) if,</u>

$$g \neq 1 \Longrightarrow (gU) \cap (U) = \phi.$$

DEFINITION 1.5 - <u>The group G acts properly on X if every point of X belongs to a proper open set.</u>

When G acts properly on X then every open set in X is the union of proper sets, so that the proper sets constitute a base for the topology of X.

It is obvious that if G acts properly on X then the action is free and if G is finite and the action is free then G acts properly on X.

We end this section with a few examples:

EXAMPLE 1.1 - Let S^1 be the unit sphere in $\mathbb{C}$ (set of complex numbers) and S^{2k+1} the unit sphere in $\mathbb{C}^{k+1}$.

The Hopf action of S^1 on S^{2k+1} is defined by:

$$\zeta(z_0,z_1,\ldots,z_k) = (\zeta z_0,\zeta z_1,\ldots,\zeta z_k), \quad (z_0,z_1,\ldots,z_k) \in S^{2k+1},\ \zeta \in S^1.$$

This action is free and the quotient space is $S^{2k+1}/S^1 = CP^k$, that is the complex projective space, which is a manifold of real dimension $2k$.

The fibration associated to this action is the Hopf fibration:

$$\begin{array}{c} S^1 \\ \downarrow \\ S^{2k+1} \\ \downarrow \\ S^{2k+1}/S^1. \end{array}$$

Since the action is free each orbit is homeomorphic to S^1.

EXAMPLE 1.2 - Let S^1 and S^{2k+1} be defined as in the previous example.

We define another action of S^1 on S^{2k+1} by:

$$\zeta(z_0,z_1,\ldots,z_k) = (\zeta^0 z_0,\zeta^1 z_1,\ldots,\zeta^k z_k).$$

This action is not free. In fact the isotropy group, G_x, of $x = (z_0,0,\ldots,0)$ is S^1; that is, x is fixed under the action of S^1. If $x = (0,z_1,0,\ldots,0)$ then $G_x = 1$, that is S^1 acts freely on this specific point x.

If $x = (0,0,\ldots, z_i,0,\ldots,0)$, $i \neq 0,1$, then $G_x = \{\zeta \mid \zeta^i = 1\}$ that is it is the set of the i-th roots of 1.

EXAMPLE 1.3 - Let S^1 be the unit circle as above and let $S^2 = \{(x,y,z) \in R^3,\ x^2 + y^2 + z^2 = 1\}$ be the unit sphere in R^3. Writing (x,y,z) as $(x + iy,z)$, we consider the action of S^1 on S^2 given by $\zeta(x + iy,z) = (\zeta(x + iy),z)$. This is a rotation about the z-axis.

This action is not free because the points $P_1 = (0,0,-1)$, $P_2 = (0,0,+1)$ are fixed. For every point $P \in S^2$, different from P_1, P_2, the isotropy group is 1, hence $0(P) = S^1$.

EXAMPLE 1.4 - Let $\mathbb{R}$ be the set of real numbers with the usual topology and $\mathbb{Z}$ the group of integers, acting on $\mathbb{R}$ by:

$$kx = x + k \qquad k \in \mathbb{Z},\ x \in \mathbb{R}$$

The open intervals of length less than 1, in $\mathbb{R}$, are proper sets. This action of $\mathbb{Z}$ is proper and the quotient space, $\mathbb{R}/\mathbb{Z}$, is homeomorphic to the unit circle S^1.

2. Morse inequalities for flows.

In this section we recall some definitions and properties of flows. For more details and proofs we refer to [6] and [7].

Suppose given a flow on a topological space Γ. This means a map $(\gamma,t) \to \gamma \cdot t$, from $\Gamma \times \mathbf{R}$ onto Γ, satisfying the following conditions:

i) $\gamma \cdot 0 = \gamma, \quad \gamma \in \Gamma, \quad 0 \in \mathbf{R}$

ii) $(\gamma \cdot s) \cdot t = \gamma \cdot (s + t), \quad \gamma \in \Gamma, \quad s,t \in \mathbf{R}$

A subset $I \subset \Gamma$ is said to be invariant if $I = \{\gamma \cdot t, t \in \mathbf{R}, \gamma \in I\} = I \cdot \mathbf{R}$.

We define the ω-limit sets of $Y \subseteq \Gamma$ as:

$$\omega(Y) = \bigcap \{c\ell(Y \cdot [t,\infty)) \mid t > 0\}$$

$$\omega^*(Y) = \bigcap \{c\ell(Y \cdot (-\infty,t]) \mid t < 0\}$$

Let I be a compact, Hausdorff, invariant set in Γ. A Morse decomposition of I is a finite collection $\{M_\pi\}_{\pi \in P}$ of disjoint, compact, invariant subsets $M_\pi \subset I$ which can be ordered $(M_1, M_2, \ldots, M_n)$ in such a way that for every $\gamma \in I \setminus \bigcup_{1 \le j \le n} M_j$ there are indices $i < j$ such that: $\omega(\gamma) \subset M_i$ and $\omega^*(\gamma) \subset M_j$. The sets M_π will be called Morse sets of I.

An ordering of $\{M_\pi\}_{\pi \in P}$ with this property will be called an admissible ordering.

A locally compact, Hausdorff subspace X of Γ is called a local flow, if for every $\gamma \in X$ there are a neighborhood $U \subset \Gamma$ of γ and an $\varepsilon > 0$ such that

$$(X \cap U) \cdot [0,\varepsilon) \subset X.$$

An invariant set S in the local flow $X \subset \Gamma$ will be called an isolated invariant set if it is the maximal invariant set in some compact neighborhood of itself. Such a neighborhood is called an isolating neighborhood for S.

It is easy to see that if $\{M_\pi\}_{\pi \in P}$ is a Morse decomposition of an isolated invariant set S, then also the sets M_π are isolated invariant sets.

A compact pair (N,N^-) will be called an index pair for the isolated invariant set S if:

a) $c\ell(N \setminus N^-)$ is an isolating neighborhood for S

b) $\gamma \in N^-$ and $\gamma \cdot [0,t] \subset N$ imply that $\gamma \cdot [0,t] \subset N^-$

c) if $\gamma \in N$, and $\gamma \cdot R^+ \not\subset N$ then there is a $t > 0$ such that $\gamma \cdot [0,t] \subset N$ and $\gamma \cdot t \in N^-$.

N^- will be also called the "exit" set of N.

It is possible to prove (see [6] and [7]) that if (N,N^-) and (N_1,N_1^-) are two index pairs for the isolated invariant set S then the pointed spaces(1) N/N^- and N_1/N_1^- are homotopically equivalent by a homotopy that moves points along orbits of the flow. Of course there exist many pairs (N,N^-) and all of these are homotopically equivalent by such a homotopy. In particular any composition of these equivalences that maps a pair to itself is homotopic to the identity map on its domain.

Thus to each S there is associated the homotopy class $[N/N^-]$ of the pointed space N/N^- obtained from an index pair, and any other pair represent[illegible] same class in a canonical way. This class will be denoted by $h(S)$ and called the (homotopy) index of S.

After these definitions we can state the Morse inequalities.

If $(M_1,\ldots,M_n)$ is an admissible ordering of a Morse decomposition of the isolated invariant set S, then:

$$\sum_{j=1}^{n} P_t(h(M_j)) = P_t(h(s)) + (1+t)Q_t \tag{2.1}$$

where $P_t(h(M_j))$ and $P_t(h(S))$ are the Poincaré series which express the Cech-cohomology (with coefficients in some fixed ring) of any element in the equivalence class $h(M_j)$ or $h(S)$, respectively, and Q_t is a series with nonnegative integer coefficients.

---

(1) If (A,B), $B \subset A$, is a topological pair then the pointed space A/B is obtained from the quotient space A/B considering the point which represents the space B as a distinguished point.

In the particular case of a smooth function $f(x)$ on a compact manifold M of dimension d from (2.1) we obtain the classical Morse inequalities.

In fact the equation:

$$\dot{x} = -\nabla f(x) \tag{2.2}$$

defines a gradient flow on M and we can take $\Gamma = M = S$.

Moreover if f has only finitely many critical points, say $C = \{x_i \mid i = 1,\dots,n\}$ the collection of the critical points, then C becomes a Morse decomposition of M by ordering its points according to the values of f.

The hypothesis that f has finitely many critical points is verified whenever f is a nondegenerate(2) function on a compact manifold.

In addition, when the critical points, x_i, are nondegenerate we have:

$$P_t(h(x_i)) = t^{d_i}$$

where d_i is the number of negative eigenvalues of the Hessian of f in the point x_i, that is it is the (Morse) index of x_i.

Regarding the (homotopy) index of $S = M$ we have:

$$P_t(h(S)) = P_t(M) = \sum_{j=0}^{d} \beta_j t^j$$

where β_j's are the Betti numbers of M, that is $P_t(M)$ is the Poincaré polynomial of M.

Finally from (2.1) we obtain:

$$\sum_{i=0}^{n} t^{d_i} = P_t(M) + (1 + t)\, Q_t(f) \tag{2.3}$$

which are the classical Morse-inequalities.

---

(2) f is nondegenerate if all its critical points are nondegenerate, that is the Hessian Hf evaluated in the critical points, never vanishes.

The polynomial $M_t(f) = \sum_{i=0}^{n} t^{d_i}$ will be called Morse polynomial of f.

Since $Q_{t(f)}$ has nonnegative coefficients the polynomial $M_t(f)$ majorizes $P_t(M)$ coefficient by coefficient. This implies that f has at least β_j critical points with index j, $j = 0,\ldots,d$.

Now we return to a general flow on Γ. As before S is an isolated invariant set in the local flow $X \subset \Gamma$.

Let M' and M'' be two Morse sets of a given decomposition. The ordered pair (M',M'') is called an adjacent pair if there is an admissible ordering $(M_1,\ldots,M_n)$ and an integer i with $M' = M_i$, $M'' = M_{i+1}$.

In this case the set $M \equiv \{x | \omega^*(x) \subset M_{i+1}, \omega(x) \subset M_i\}$ is also an isolated invariant set and the collection $(M_1,\ldots M_{i-1},M,M_{i+2}\ldots M_n)$ is a "coarser" Morse decomposition. Furthermore, there is a canonical exact sequence

$$\ldots \xleftarrow{\delta} H_*(h(M_i)) \leftarrow H_*(h(M)) \leftarrow H_*(h(M_{i+1}) \xleftarrow{\delta} \ldots.$$

If the connecting homomorphism, δ, of this sequence is non-trivial then $M \neq M' \cup M''$ - i.e. M'' must be 'connected' to M' by an orbit of the flow.

On the other hand if all such connecting homomorphisms for all adjacent pairs (in any admissible ordering) are trivial, then the decomposition is "perfect" in the following sense:

DEFINITION 2.1 - A Morse decomposition $(M_1,\ldots,M_n)$ of S is said to be K-perfect if relation (2.1) holds with $Q_t = 0$, when the cohomologies are taken with coefficients in K.

We will not indicate the dependence on K, when the Morse decomposition is perfect with respect to any coefficient ring K.

When we have the gradient-flow (2.2) given by a nondegenerate function f, we will call this function (K-) perfect if (2.3) holds with $Q_t(f) = 0$.

An important consequence of Definition 2.1 is that whenever we have a perfect Morse decomposition of S then we can have information about the (homotopy) index of S by looking at the left hand side of (2.1).

In the case of a gradient flow on the compact manifold M this means that we can compute the cohomology of M by the computation of the Morse polynomial of any perfect nondegenerate function f defined on M.

A criterion to recognize a perfect Morse decomposition of S is the following Morse's lacunary principle which follows immediately from (2.1):

<u>If, taking some ring of coefficients K, no consecutive powers of t occur in the left hand side of (2.1), then</u> $Q_t = 0$ <u>so that:</u>

$$\sum_{j=1}^{n} P_t(h(M_j)) = P_t(h(S)) \tag{2.4}$$

Some examples about the use of this principle will be furnished in section 4.

3. Equivariant flows.

In this section we suppose that there is a (left) action of a group G on the topological space Γ and that the isolated invariant set S is invariant under this action (that is $g\gamma \in S$, if $g \in G$ and $\gamma \in S$). When this happens we say that S is G-invariant, to distinguish this property from the invariance with respect to the flow.

We say that the flow on Γ is equivariant if:

$$(g\gamma) \cdot t = g(\gamma \cdot t) \qquad \gamma \in \Gamma, g \in G, t \in \mathbb{R}. \tag{3.1}$$

If we have a gradient flow (2.2) on a G-invariant compact manifold M then it is equivariant if the function f is G-invariant, that is if: $f(gx) = f(x)$, $x \in M$, $g \in G$.

From now on we will restrict our attention to the isolated invariant set S.

In order to study an equivariant flow, the most natural thing would be to look at the quotient space S/G.

In fact it is obvious that we can define a flow on S/G in the following way:

$$[\gamma] \cdot t = [\gamma \cdot t] \qquad [\gamma] \in S/G,\ t \in \mathbb{R} \tag{3.2}$$

where $[\gamma]$ is the orbit (equivalence class) of γ under the action of G.

The flow (3.2) is well-defined because if γ and γ' belong to the same equivalence class then: $\gamma' = g\gamma$, for some $g \in G$ and consequently:

$$[\gamma'] \cdot t = [\gamma' \cdot t] = [(g\gamma) \cdot t] = [g(\gamma \cdot t)] = [\gamma \cdot t] = [\gamma] \cdot t$$

If I is a G-invariant isolated invariant set in S, then I/G is an isolated invariant set in S/G.

Moreover it is possible to find an index pair (N,N^-) of I, with N and N^- G-invariant and such that the pair $(N/G, N^-/G)$ is an index pair for I/G.

Finally if $(M_1,\ldots,M_n)$ is an admissible ordering of a Morse decomposition of S, given by G-invariant Morse sets, then $(M_1/G,\ldots,M_n/G)$ is an admissible ordering of a Morse decomposition of S/G.

A second approach to the study of an equivariant flow would be to look at the isolated invariant set S but considering Morse decompositions whose Morse sets contain

the complete G-orbit of any point in the set (these orbits may be topologically different, in general).

In this connection we note that if I is an isolated invariant set and $O(I) = \{g\gamma, g \in g, \gamma \in I\}$ is the orbit of I, then $O(I)$ is also an isolated invariant set and, if the group G is continuous, $O(I) = I$.

A third approach is the "equivariant theory" which consists of extending the flow to the space $S \times E$, where E is a contractible space on which G acts freely, and then obtaining the Morse inequalities in the quotient space $(S \times E)/G$, replacing the cohomology of the spaces involved in (2.1) with their "equivariant cohomology".

We will explain the equivariant theory in detail in the next section.

When the action of G on S is free then there is not much difference between these three methods; in particular the first and the third one give exactly the same answer because, in this case, the equivariant cohomology coincides with the ordinary cohomology.

When the action is not free, then in general each approach furnishes different information; that is, the Morse inequalities provide different consistency conditions.

To understand this difference it is enough to think about the difference between S, S/G, $(S \times E)/G$ at the cohomological level.

It may happen that a space X has a trivial cohomology (which does not give much information) but X/G has a rich cohomology and vice versa.

For instance if S^∞ is the sphere in an Hilbert space and S^1 acts on it with the Hopf action, then $P_t(S^\infty) = 1$ because S^∞ is contractible while $P_t(S^\infty/S^1) = P_t(CP)^\infty = 1 + t^2 + t^{2n} + \ldots = \dfrac{1}{1 - t^2}$.

Moreover if we have a gradient-flow on a compact G-invariant manifold M given by a G-invariant nondegenerate function f and if the action on M is not free then the classical Morse theory does not apply because M/G is not, in general, a manifold. The more general approach described here does apply, but gives different information from the equivariant theory. Thus, in this case, it is reasonable to use the equivariant theory which is a natural extension of the free case.

We will support what we have claimed so far with examples in the next section.

We end, giving the definition of nondegenerate critical manifold for a smooth function f on a compact d-dimensional manifold M and characterizing its Morse index.

We say that a connected submanifold $T \subset M$ is an isolated critical manifold if:

i) each point $p \in T$ is a critical point of f

ii) T is isolated as a critical point set

From i) and ii) it follows that T is an isolated invariant set in the gradient-flow (2.2). Then T has an homotopy-index $h(T)$ as always. This can be computed as follows in the case where T is "non-degenerate."

Namely, if the critical manifold T satisfies i) and :

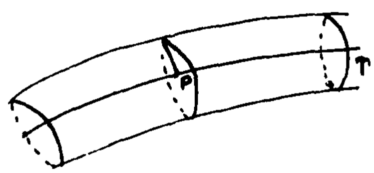
ii)' the Hessian of f is nondegenerate in the normal direction to T,

then we say that T is a *nondegenerate critical manifold.*

ii)' means that if $(x_1,\ldots,x_k,x_{k+1}\ldots,x_d)$ is a system of local coordinates in M, centered at p, such that near p, T is given by the $n - k$ equations: $x_{k+1} = 0,\ldots,x_n = 0$, then

$$\det \left| \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right) \right|_p \neq 0 \qquad \text{for } i,j = k+1,\ldots,n$$

Another way of saying this, is considering a small tubular ε-neighborhood $M_\varepsilon(T)$ fibered over T by the normal discs to T, relative to some Riemann structure on M. Thus ii)' means that f restricted to each normal disc is nondegenerate.



Moreover ii)' implies ii), that is each nondegenerate critical manifold is also isolated.

We denote by $\nu(T)$ the normal bundle of T endowed with a Riemannian metric and by $H_T f$ the Hessian of f on $\nu(T)$.

If we set:

$$(A_T x,y) = H_T f(x,y) \qquad x,y \in \nu(T)$$

then we define a self adjoint endomorphism from $\nu(T)$ to $\nu(T)$.

Hypothesis ii)' implies that A_T does not have zero as an eigenvalue and hence $\nu(T)$ can be decomposed into the direct sum:

$$\nu(T) = \nu^+(T) \oplus \nu^-(T)$$

where $\nu^+(T)$ and $\nu^-(T)$ are spanned (respectively) by the positive and negative eigenvalues of A_T.

The fiber dimension, λ_T, of $\nu^-(T)$ will be called the index of T as a critical manifold of f. Now we want to write the Morse inequalities (2.1) in the case of a smooth function f whose critical sets are only nondegenerate critical manifolds.

The contribution in the Morse polynomial of a critical manifold T is:

$$M_t(T) = \sum_i t^i \text{ rank } H_c^i \{\nu^-(T)\} \tag{3.3}$$

where H_c^i denotes the compactly supported cohomology[3] (see [16]).

At this point it is better to remark that, in the nondegenerate case, $M_t(T)$ is equal to $P_t(h(T))$ because the "exit" directions in $M_\varepsilon(T)$ are those of $\nu^-(T)$ and

---

(3) If X is a locally compact topological space:

$$H_c^i(X) = H^i(\hat{X}) \qquad i = 1,2,\ldots$$

where $\hat{X}$ is the one point-compactification of X

Ex: $H_c^n(\mathbb{R}^n) = K$ (if K is the coefficient ring)

the compactly supported chomology of $\nu^-(T)$ is the cohomology of N/N^-, N being an isolating neighborhood of T and N^- its "exit" set.

By the Thom isomorphism:

$$H_c^i\{\nu^-(T)\} = H^{i-\lambda_T}(T,\theta^-\otimes K)$$

where K is a ring, θ^- is the orientation bundle of $\nu^-(T)$ and $H^*(T,\theta^-\otimes K)$ is the cohomology with local coefficients.

Hence (3.3) becomes:

(3.4) $$M_t(T) = t^{\lambda_T}P_t(T,\theta^-\otimes K)$$

In particular, when the bundle $\nu^-(T)$ is orientable[4] $P_t(T,\theta^-\otimes K) = P_t(T,K)$. Then, if we consider a Morse decomposition of M given by the nondegenerate critical manifolds of f, (2.1) becomes:

(3.5) $$\sum_T t^{\lambda_T}P_t(T,\theta^-\otimes K) = P_t(M) + (1+t)Q_t$$

In (3.5) it is understood that the sum is taken over all the critical manifolds of f.

---

(4) We say that a fibration

$$\begin{array}{c} F \\ \downarrow \\ V \\ \downarrow p \\ B \end{array}$$

is orientable over a ring K is for any closed path ω in B, with $\omega(0) = \omega(1) = b \in B$, the induced map:

$$\tau_\omega^*:\ H^*(F_b;K) \to H^*(F_b;K)$$

is the identity.

In particular, if B is simply connected every fibration over B is orientable, over any K.

4. Equivariant Morse theory.

We assume, as in the previous section, that an equivariant flow on Γ is defined and S is G-invariant, G being a topological group acting on Γ.

If G is compact then (see [9]) there is an universal G-bundle characterized by having its total space E contractible:

$$
\begin{array}{c}
G \\
\downarrow \\
E \\
\downarrow \\
E/G = BG
\end{array}
\tag{4.1}
$$

The space BG is called the classifying space of G.

The action of G on E is free and E is unique, up to homotopy.

Since the action of G on E is free, the diagonal action of G on the product $S \times E$ is free too.

Here diagonal action means:

$$g(\gamma,e) = (g\gamma,ge) \qquad g \in G, \ \gamma \in S, \ e \in E$$

We can extend the flow to $S \times E$ in the trivial way:

$$(\gamma,e) \cdot t = (\gamma \cdot t,e) \quad t \in R.$$

As shown in section 3 we can project this flow on the quotient space $(S \times E)/G = S_G$

It is obvious that if I is a G-invariant, invariant set for the flow on S then $(I \times E)/G = I_G$ is an invariant set for the quotient-flow in S_G.

Our aim is to obtain the analogue of the Morse inequalities (2.1) for this quotient flow using the equivariant cohomology.

To do this we need some compactness condition. In fact in obtaining (2.1) compact pairs have been used. Also the definition of isolated invariant set requires the presence of a compact isolating neighborhood.

But in the bundle (4.1), usually, E and BG are realized as infinite dimensional

manifold, so all compactness is lost in $S \times E$ and S_G. This difficulty can be overcome in the following way.

When G is a compact topological group, E and BG can be obtained as limit of finite dimensional compact spaces:

$$E = \lim_{k\to\infty} E_k \qquad BG = \lim_{k\to\infty} B_k G$$

related to the bundles:

$$\begin{array}{c} G \\ \downarrow \\ E_k \\ \downarrow \\ E_k/G = B_k G. \end{array}$$

The action of G on E_k is free.

So the Morse-inequalities are obtained for each k and we pass to the limit using the stabilizing properties of cohomology.

If $\{M_1,\ldots,M_n\}$ is an admissible ordering of a Morse decomposition of S and each M_j is G-invariant then:

$$\{(M_1 \times E_k)/G,\ldots,(M_n \times E_k)/G\}$$

is a Morse decomposition for the isolated invariant set $(S \times E_k)/G$. Observe that the flow in $S \times E_K$ is defined in the trivial way, as for $S \times E$.

Also if (N,N^-) is an index pair with N and N^- G-invariant for the G-invariant isolated invariant set I then

$$((N \times E_k)/G,(N^- \times E_k)/G) = (N_k,N_k^-)$$

is an index pair for $(I \times E_k)/G$.

So if we denote by $h_k(I)$ the (homotopy) index associated to any index pair (N_k,N_k^-) of $(I \times E_k)/G$, we obtain:

$$(4.2) \qquad \sum_{j=1}^{n} P_t(h_k(M_j)) = P_t(h_k(S)) + (1 + t)Q_t^k \qquad k = 1,2,\ldots.$$

Now we pass to the limit in (4.2), for $k \to \infty$, using the stabilization of the cohomology for the classifying space, (see [9] Chap. III) that is: for $E = \lim E_k$ and $BG = \lim E_k/G$, then: for each $i \in N$, there exists $m(i) \in N$, such that

$$k > m(i) \Longrightarrow H^i(E) \cong H^i(E_k) \quad \text{and} \quad H^i(BG) \cong H^i(E_k/G).$$

Hence we obtain:

$$(4.3) \qquad \sum_{j=1}^{n} P_t^G\,(h(M_j)) = P_t^G\,(h(S)) + (1+t)Q_t^G$$

where the Poincaré series $P_t^G(h(S))$ (resp. $P_t^G(h(M_j))$) represents the cohomology of the pair $\left((N \times E)/G, (N^- \times E)/G\right)$, if (N,N^-) is a G-invariant index pair for S (resp. for M_j), that is the equivariant chomology of (N,N^-).[5]

The homotopy type of the pair $\left((N \times E)/G, (N^- \times E)/G\right)$, will be denoted by $h_G(I)$ and called the equivariant-(homotopy) index of I.

With this understood (4.3) becomes:

$$(4.4) \qquad \sum_{j=1}^{n} P_t(h_G(M_j)) = P_t(h_G(S)) + (1+t)Q_t^G.$$

---

(5) If G acts on a space X and E is defined by (4.1) then the equivariant cohomology of X, $H^*_G(X)$, is:

$$H^*_G(X) = H^*(X_G)$$

where $X_G = (X \times E)/G$.

If $X = \{x_0\}$ then $H^*_G(x_0) = H^*(BG)$, that is $H^*(BG)$ is the equivariant cohomology of a point.

If G acts freely on X, then the map:

$$p\colon\ X_G \dashrightarrow X/G \qquad p([(x,e)]) = [x]$$

is a homotopy equivalence.

Hence

$$H^*_G(X) \cong H^*(X/G)$$

that is the equivariant cohomology of X is the cohomology of the quotient space X/G.

If G is a compact Lie group and we have the gradient flow induced by a nondegenerate G-invariant smooth function f on the G-invariant compact manifold M (4.3) can be written in a more explicit way.

First of all associated to the bundle (4.1) there is another one:

$$
\begin{array}{c}
M \\
\downarrow \\
(M \times E)/G = M_G \\
\downarrow \\
BG
\end{array}
\tag{4.5}
$$

Observe that since G acts freely on $M \times E$, M_G is a manifold.

Then it is easy to see that f can be lifted to a G-invariant function on $M \times E$ and hence projected to a function f_E on M_G.

The most important thing is that f_E is still a nondegenerate function as it is shown in the next Proposition (see [1]).

Proposition 4.1 - _If f is a nondegenerate function on M, then for every smooth principal G-bundle E, f_E is nondegenerate on M_G. Moreover, if N is a nondegenerate critical manifold of f on M, then f_E will have as corresponding critical manifold the space $(N \times E)/G$. Finally, the Morse indices of N relative to f and $(N \times E)/G$ relative to f_E are equal._

This Proposition suggests writing the Morse inequalities for the nondegenerate function f_E. Of course, since M_G is not compact, this can be done using the same finite dimensional-approximation method used to obtain (4.3).

Then, from (3.5) and (4.3) we have:

$$
\sum_T t^{\lambda_T} P_t^G(T, \theta^- \otimes K) = P_t^G(M) + (1 + t)Q_t^G
\tag{4.6}
$$

where $P_t^G(M) = P_t(M_G)$ and $P_t^G(T) = P_t\big((T \times E)/G\big)$, T being a critical manifold of f.

In particular if T consists of a single orbit: $T = G/H$, where H is the isotropy group of each point of T, we have:

$$(T \times E)/G = (G/H \times E)/G \cong E/H.$$

But, since E is an universal G-bundle, E is also an universal H-bundle. This implies that E/H is homotopically equivalent to BH, the classifying space of H.

Then, in this case:

$$t^{\lambda_T} P_t^G(T, \theta^- \otimes K) = t^{\lambda_T} P_t(BH; \theta^- \otimes K). \tag{4.7}$$

Furthermore, if H is connected $P_t(BH, \theta^- \otimes K) = P_t(BH, K)$[6], otherwise local coefficients may be needed.

Having defined the equivariant Morse theory, now we are ready to illustrate, through some examples the difference between the three ways of studying an equivariant flow, described in section 3.

EXAMPLE 4.1 - Consider the free action of S^1 on S^{2k+1} defined in Example 1.1 and the function:

$$f(z_0, z_1, \dots, z_k) = \sum_0^k{}_i \lambda_i |z_i|^2$$

where $\lambda_0 < \lambda_1 < \dots < \lambda_k$ are a sequence of distinct real numbers.

---

(6) From the fibration:

$$\begin{array}{c} H \\ \downarrow \\ E \\ \downarrow \\ BH \end{array}$$

We have the following exact homotopy sequence:

$$\dots \to \pi_1(H) \to \pi_1(E) \to \pi_1(BH) \to \pi_0(H) \to \dots$$

where $\pi_i(\ )$ is the ith homotopy group.

Since E is contractible and H is connected $\pi_1(BH)$ is trivial, that is BH is simply connected.

Then from note (4) the bundle $\nu^-(BH)$ is orientable.

The function f is invariant under the action of S^1 and so it defines a function, which we will continue to denote by f, on the quotient space CP^k.

Using the principle of Lagrange multiplier, for example, we can see that the critical points of f correspond to the complex coordinate axes. The eigenvalues of the Hessian of f along the ith axis are the numbers:

$$\lambda_0 - \lambda_i, \ldots, \lambda_{i-1} - \lambda_i, \lambda_{i+1} - \lambda_i, \ldots, \lambda_k - \lambda_i$$

so that exactly i are negative.

Since over the reals their multiplicity is 2 the index of the ith critical point is $2i$.

Hence we have:

$$M_t(f) = 1 + t^2 + \ldots + t^{2k}$$

and since there are no consecutive powers the Lacunary Morse principles (2.4) applies, giving:

$$P_t(CP^k) = 1 + t^2 + \ldots + t^{2k}$$

that is the cohomology of the complex projective space, with any coefficients field. Thus, studying the gradient flow on the quotient space we have obtained a perfect function.

If we had studied the flow on S^{2k+1} then, considering that each critical point gives rise to an S^1 critical orbit, we would have obtained:

$$(1 + t)(1 + t^2 + \ldots + t^{2k}) = 1 + t^{2k+1} + (1 + t)(t + t^3 + \ldots + t^{2k-1})$$

where $1 + t^{2k+1} = P_t(S^{2k+1})$. This means that f is not perfect on S^{2k+1}.

Before considering the next example we want to remark that, actually, if S is an isolated invariant set in a local flow, two (homotopy) indexes are defined, according to the two directions of the time.

The first one, in the forward direction is the one already defined. The second, in backward time, can be defined "reversing" the flow with respect to the time. This means

that we consider an index-pair (N,N^+) where N^+, the "entrance" set, is defined by the properties dual with respect to those which define N^-.

In the gradient-flow case this is realized by considering $-f$ instead of f. Consequently, considering a Morse decomposition of S, we have two different kinds of Morse inequalities, according to the two different indexes of the Morse sets.

This, in general, gives more information. For example, suppose the isolated invariant set S is the total space. Then the indexes in the two different directions are the same. Now if the Poincaré polynomial $P_t(h(S))$ is not symmetric(7), different information comes from the two sets of Morse relations.

Of course, if M is a compact manifold (without boundary) then, from the Poincaré duality Theorem(8), its Poincaré polynomial is symmetric, but, since the Morse theory applies also to manifolds with boundary (or general compact metric spaces) the consideration of the index in both directions can be really useful.

This happens, in particular, when we have a quotient space M/G, where M is a manifold and G does not act freely on M, as we will see in the next examples.

EXAMPLE 4.2 - Let us consider the action of S^1 on S^2 defined in Example 1.3 and the function

$$f(x,y,z) = z \quad \text{on } S^2.$$

The only two critical points of f are the two fixed points P_1 and P_2 (resp. min. and max. of f).

Let us examine the three different approaches:

a) First of all we consider the quotient space S^2/S^1 which is homeomorphic to the

(7) A polynomial: $a_0 + a_1t + \ldots + a_nt^n$ is symmetric if $a_i \neq 0 \Rightarrow a_{n-i} \neq 0$.

(8) The Poincaré duality Theorem essentially claims that if M is an n-dimensional compact manifold and K is a field then $H^i(M,K)$ is isomorphic to $H^{n-i}(M,K)$.

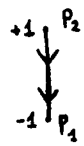
interval $[-1,1]$ on the z-axis. This is a contractible set and hence has a trivial cohomology:

$$P_t(S^2/S^1) = 1$$

So it seems that from this cohomology we can just guess the presence of one critical point, that is the minimum of f.

But if we reverse the flow with respect to the time direction, that is, if we consider $-f$ instead of f we discover another critical point. In fact, since $P_t(S^2/S^1)$ does not change also $-f$ has to have a minimum that cannot be the same as the one of f.

On the other hand we know that there are two critical points and that one is an attractor and one is a repeller for the gradient flow on S^2/S^1:



The (homotopy)-indexes are:

$$h(P_1) = \bar{1} \quad \text{and} \quad h(P_2) = \bar{0}$$

that is $h(P_1)$ corresponds to the homotopy type of the pointed 0-sphere and $h(P_2)$ corresponds to the homotopy type of the pointed one-point space. (see [6])

Hence, considering the Morse decomposition (P_2,P_1) we have:

$$P_t(h(P_2)) + P_t(h(P_1)) = 1 + 0 = P_t(S^2/S^1) = 1.$$

Consequently (P_2,P_1) is a perfect Morse decomposition of S^2/S^1.

b) Here we consider the function directly on S^2 and we look at the critical orbits. These consist of P_1 and P_2, since these two points are fixed under the action of S^1.

The Morse indexes, as number of negative eigenvalues, of P_1 and P_2 are 0 and 2, respectively. The cohomology of S^2 is: $P_t(S^2) = 1 + t^2$. Then we have:

$$M_t(f) = 1 + t^2 = P_t(S^2) = 1 + t^2$$

that is f is still a perfect function.

c) Finally we use the equivariant approach. Looking at the fibration:

$$\begin{array}{c} S^2 \\ \downarrow \\ (S^2 \times E)/S^1 \\ \downarrow \\ BS^1 \end{array} \tag{4.8}$$

E being the total space of an universal bundle of S^1, we have:

$$P_t^{S^1}(S^2) = P_t(S^2) \cdot P_t(BS^1) = \frac{1 + t^2}{1 - t^2}\,. \tag{4.9}$$

We have obtained the product formula (4.9) from the spectral sequence associated to (4.8), observing that the classifying space of S^1 is the infinite-dimensional complex projective space whose cohomology is $1 + t^2 + \ldots + t^{2n} + \ldots$.

Since every critical manifold of f consists of a single orbit (*namely* P_1 *or* P_2) and the isotropy group is S^1, we can apply (4.7), with $BH = BS^1$.

Then we obtain

$$M_t^{S^1}(f) = \frac{1}{1 - t^2} + \frac{t^2}{1 - t^2} = P_t^{S^1}(S^2) = \frac{1 + t^2}{1 - t^2}$$

Hence f is equivariantly perfect.

Let us observe that in this case, as in the previous one, reversing the flow nothing changes.

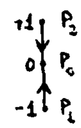
EXAMPLE 4.3 - We consider the same action as in the previous example and the function:

$$f(x,y,z) = z^2 \quad \text{on} \quad S^2.$$

This function has a minimum corresponding to the circle orbit at $z = 0$ and two maxima corresponding to the points with $z = \pm 1$.

We have:

a) In the quotient space S^2/S^1 the point P_0 with $z = 0$ is an attractor, $P_t(h(P_0)) = 1$, P_1 and P_2 are both repellers, $P_t(h(P_j)) = 0$, $j = 1,2$.



Hence, considering the Morse decomposition (P_1, P_2, P_0) we obtain:

$$\sum_{j=0}^{2} P_t(h(P_j)) = 1 = P_t(S^2/S^1)$$

that is the Morse decomposition is perfect.

If we reverse the flow, then P_0 becomes a repeller and P_1, P_2, both attractors. The associated Morse decomposition is (P_0, P_1, P_2) and we have: $P_t(h(P_0)) = t$, $P_t(h(P_j)) = 1$, $j = 1,2$.

Thus the Morse inequalities are:

$$\sum_{j=0}^{2} P_t(h(P_j)) = t + 2 = P_t(S^2/S^1) + 1 + t$$

Therefore (P_0, P_1, P_2) is not a perfect Morse decomposition.

b) Considering f on S^2 we have a critical orbit homeomorphic to S^1 corresponding to the minimum whose contribution in the Morse inequalities, according to (3.4) is:

$$t^0 P_t(S^1) = 1 + t.$$

The other two critical orbits are the points P_1 and P_2 whose Morse index is 2 (nondegenerate maxima).

So we have:

$$M_t(f) = (1 + t) + 2t^2 = P_t(S^2) + (1 + t)Q_t = 1 + t^2 + (1 + t)t$$

This means that, using this approach, f is not perfect and $Q_t = t$.

Reversing the flow we have:

$$M_t(f) = 2 + t(1 + t) = P_t(S^2) + 1 + t$$

that is f is still not perfect and $Q_t = 1$.

c) Using the equivariant theory and considering that the isotropy group of each point of the circle orbit is $\{1\}$, (1 is the unity element in the group S^1), we get:

$$M_t^{S^1}(f) = 1 + \frac{2t^2}{1 - t^2} = \frac{1 + t^2}{1 - t^2} = P_t^{S^1}(S^2)$$

hence f is perfect.

If we reverse the flow, then we have:

$$M_t^{S^1}(f) = \frac{2}{1 - t^2} + t = \frac{1 + t^2}{1 - t^2} + (1 + t)$$

Consequently f is not perfect and $Q_t = 1$.

EXAMPLE 4.4- Let S^{2n} be the unit sphere in $R^{2n+1} \cong C^n \times R$.

A point in S^{2n} will be denoted by:

$$z = (z_1, \ldots, z_n, x) \quad z_i \in C,\ x \in R$$

We consider the action of S^1 on S^{2n} defined by:

(4.10) $\qquad \zeta z = (\zeta z_1, \ldots, \zeta z_n, x) \qquad \zeta \in S^1$

This action leaves the x-axis fixed and induces on the "equator" = $= \{z = (z_1, \ldots, z_n, 0)\}$ the Hopf action of Example 1.1.

Then we consider the function:

$$f(z) = x \quad \text{on} \quad S^{2n}$$

The two critical points of f are:

$$\underline{z} = (0, \ldots, 0, -1)\ , \quad \bar{z} = (0, \ldots, 0, +1)$$

Our aim is to use the Morse lacunary principle to find the cohomology of the quotient space S^{2n}/S^1.

To do this we need to compute the homotopy-indexes of $\underline{z}$ and $\bar{z}$.

Because $\underline{z}$ is an attractor (actually it is the minimum) it is easy to see that $h(\underline{z}) = \bar{1}$ and hence $P_t(h(\underline{z})) = 1$.

To compute the index of $\bar{z}$ we can consider the index pair $(B/S^1, B^-/S^1)$ defined in the following way: B is the compact neighborhood of $\bar{z}$ given by $B = \{ (z_1,\ldots,z_n,x) \in S^{2n},\ 0 < \varepsilon < x < 1\}$ and B^- is its boundary.

We can compute the cohomology of $(B/S^1, B^-/S^1)$ from the following exact sequence:

$$(4.11)\quad 0 \to H^0(B/S^1,B^-/S^1) \to H^0(B/S^1) \to H^0(B^-/S^1) \xrightarrow{\delta^0} H^1(B/S^1,B^-/S^1) \to \ldots \to$$

$$\ldots \to H^i(B^-/S^1) \xrightarrow{\delta^i} H^i(B/S^1,B^-/S^1) \to H^i(B/S^1) \xrightarrow{\delta^{i+1}} \ldots$$

where δ^i is the coboundary operator.

The cohomology of B/S^1 is: $P_t(B/S_1) = 1$ since B/S^1 is contractible. The cohomology of B^-/S^1 is: $P_t(B^-/S^1) = 1 + t^2 + \ldots + t^{2n-2}$ because B^-/S^1 is homeomorphic to the complex projective space.

Then, from (4.11):

$$P_t(h(\bar{z})) = P_t(B/S^1,B^-/S^1) = 1 + t^3 + t^5 + \ldots + t^{2n-1}$$

Putting this together with $P_t(h(\underline{z}))$ we have that the left hand side of the Morse inequalities on S^{2n}/S^1 are:

$$1 + t^3 + t^5 + \ldots + t^{2n-1}.$$

Hence, since no consecutive powers occur, f is a perfect function on the quotient space and the homology of S^{2n}/S^1 is:

$$P_t(S^{2n}/S^1) = 1 + t^3 + t^5 + \ldots + t^{2n-1}.$$

In the last example we want to show how, sometimes, the presence of critical points can be deduced just from the properties of the group action.

EXAMPLE 4.5 - Let S^{2n-1} be the unit sphere in C^n. We define an action of S^1 on S^{2n-1} by:

$$\zeta z = \zeta(z_1,\ldots,z_n) = (\zeta z_1,\zeta^2 z_2,\ldots,\zeta^n z_n) \qquad \zeta \in S^1,\ z \in S^{2n-1}$$

It is easy to see that there are no fixed points. On the other hand the action is not free.

In fact the point $z = (0,\dots,z_i,0,\dots,0)$ $(1 < i < n)$ has isotropy group $\mathbb{Z}_i$.

For $1 < p < n$, we define the following sets:

$$X_p = \{ z \in S^{2n-1} \text{ such that } G_z \supseteq \mathbb{Z}_p \}$$

where G_z is the isotropy group of z.

Now, suppose that an S^1-equivariant flow is defined on S^{2n-1}. Then each X_p is a compact invariant set for the flow.

In fact, if $z \in X_p$ and $t \in \mathbb{R}$ we have:

$$\zeta(z \cdot t) = (\zeta z) \cdot t = z \cdot t, \quad \text{for } \zeta \in \mathbb{Z}_p.$$

Then, since

$$X_p = \{ z = (z_1,\dots,z_n) \in S^{2n-1} \text{ such that } z_i = 0,\ n > i \neq kp,\ k \in \mathbb{N} \}$$

X_p is a closed subset of S^{2n-1} and hence it is compact.

Therefore if our flow is gradient-like[(8)'] each set X_p has to contain at least one orbit of "rest" points[(8)"], that is points $\bar{z}$ such that $\bar{z} \cdot \mathbb{R} = \bar{z}$.

In particular if the flow is a gradient-flow given by a smooth (but not necessarily nondegenerate) function f defined on S^{2n-1}, each X_p contains an orbit of critical points of f.

This implies that any function f on S^{2n-1}, which is S^1-invariant, with respect to this action has at least $n - 2$ critical orbits.

REMARK 4.1 - If, instead of considering the S^1 action of the previous example, we

---

(8)' A flow is gradient-like if there is a continuous real valued function which is strictly decreasing on the nonconstant orbits of the flow. Such a function is called a Liapunov function.

---

(8)" Actually there are two distinct orbits (corresponding to the extrema of the Liapunov function) as soon as X_p is not just one orbit, that is, X_p is a sphere of dimension greater than 1.

had considered the action of Example 1.2 of Section 1, then the same argument would have been true.

But, because of the presence of fixed points, the intersection of the sets X_p is just the set of the fixed points which also contains two critical points.

Then in this case, we could not have deduced, from the previous argument, the presence of more than two critical points.

An application of Morse theory to the study of a function on a finite dimensional sphere in presence of a finite symmetry group is the following theorem (see [2]).

THEOREM 4.1 - Let f be a G-invariant smooth function on the sphere $S^n \subset R^{n+1}$, where G is a finite group.

If G acts on S^n without fixed points then f has at least $n + 1$ *orbits of* critical points.

This theorem has been applied in [2] to obtain a multiplicity result in a bifurcation problem with symmetry.

Another application of the equivariant Morse theory to the N-body problem can be found in [14].

5. Cohomology of groups.

Let R be a ring with identity 1, and C a (left) R-module. A resolution over C is an exact sequence of R-modules:

$$(5.1) \qquad \cdots \to X_n \xrightarrow{\partial_n} X_{n-1} \xrightarrow{\partial_{n-1}} \cdots \longrightarrow X_1 \xrightarrow{\partial_1} X_0 \xrightarrow{\varepsilon} C \longrightarrow 0$$

The resolution is called projective if every X_n is projective,[9] free, if any X_n is free.

A resolution over C will be denoted by $(X \overset{\varepsilon}{\to} C)$.

A free R-module C admits always the free resolution:

$$0 \to C \xrightarrow{id} C \to 0.$$

Any R-module C is a quotient, $C = F^0/R_0$ of some free R-module F_0. The submodule R_0 is again a quotient $R_0 = F_1/R_1$ of a free module F_1.

Continuing this process we have the free-resolution over C:

$$\cdots \to F_1 \to F_0 \to C \to 0.$$

For example if $C = \mathbb{Z}_2$ is considered as $\mathbb{Z}$-module, $\mathbb{Z}_2 = \frac{\mathbb{Z}}{2\mathbb{Z}}$ then the following resolution is free:

$$0 \to 2\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0.$$

Let A be a fixed R-module. We apply the controvariant functor $\mathrm{Hom}_R(-,A)$ to any resolution over C.

Since this functor does not preserve exactness, the resulting sequence may not be exact:

$$(5.2) \qquad 0 \longrightarrow \mathrm{Hom}_R(C,A) \xrightarrow{\varepsilon^*} \mathrm{Hom}_R(X_0,A) \xrightarrow{\delta_0} \mathrm{Hom}_R(X_1,A) \xrightarrow{\delta_1} \cdots$$

---

(9) An R-module C is projective if given an epimorphism β from the R-module B to C and an homomorphism γ from the R-module A to C then another homomorphism γ can be found such that the following diagram is commutative:

$$\begin{tikzcd} & A \arrow[dl, dashed, "\gamma"'] \arrow[d, "\alpha"] \\ B \arrow[r, two heads, "\beta"'] & C \end{tikzcd}$$

Every free R-module is projective.

This implies that the cohomology $H^n(X,A) = H^n(\mathrm{Hom}_R(X,A))$ is not trivial, in general.

We want to prove that $H^n(X,A)$ depends only on C and A and not on the particular projective resolution choosen.

We need a Lemma:

LEMMA 5.1 - Suppose that $\gamma: C \dashrightarrow C'$ is an homomorphism of R-modules, $(X \xrightarrow{\varepsilon} C)$ and $(X' \xrightarrow{\varepsilon'} C')$ are two projective resolutions over C and $h: X \dashrightarrow X'$ is a chain transformation[10] with the property

$$\varepsilon' h_0 = \gamma\varepsilon.$$

If there exists $t: C \to X_0'$ such that $\varepsilon' t = \gamma$, then there are homomorphisms s_n: $X_n \to X_{n+1}'$, $n = 0,1,\ldots,$ such that:

(5.3) $$\partial_1' s_0 + t\varepsilon = h_0, \quad \partial_{n+2}' s_{n+1} + s_n \partial_{n+1} = h_{n+1} \qquad \forall n.$$

Proof. We have the following commutative diagram:

(5.4)
$$\begin{tikzcd}
\cdots \arrow[r] & X_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "h_{n+1}"] & X_n \arrow[r, "\partial_n"] \arrow[d, "h_n"] & X_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "h_{n-1}"] & X_{n-2} \arrow[r] \arrow[d, "h_{n-2}"] & \cdots \arrow[r] & X_1 \arrow[r, "\partial_1"] \arrow[d, "h_1"] & X_0 \arrow[r, "\varepsilon"] \arrow[d, "h_0"] & C \arrow[r] \arrow[d, "\gamma"] \arrow[dl, dashed, "t"'] & 0 \\
\cdots \arrow[r] & X_{n+1}' \arrow[r, "\partial_{n+1}'"] & X_n' \arrow[r, "\partial_n'"] & X_{n-1}' \arrow[r, "\partial_{n-1}'"] & X_{n-2}' \arrow[r] & \cdots \arrow[r] & X_1' \arrow[r, "\partial_1'"] & X_0' \arrow[r, "\varepsilon'"] & C' \arrow[r] & 0
\end{tikzcd}$$

---

(10) A chain transformation f from $X = \{X_0, X_1, \ldots, X_n \ldots\}$ to $X' = \{X_0', \ldots, X_n', \ldots\}$ is a family of module-homomorphisms: $f_n: X_n \to X_n'$ such that:

$$\partial_n' f_n = f_{n-1}\partial_n \qquad \forall n.$$

where ∂_n' and ∂_n are module homomorphisms:

$$\dashrightarrow X_n' \xrightarrow{\partial_n'} X_{n-1}' \xrightarrow{\partial_{n-1}'} X_{n-2}' \dashrightarrow \ldots; \quad \dashrightarrow X_n \xrightarrow{\partial_n} X_{n-1} \xrightarrow{\partial_{n-1}} X_{n-2} \dashrightarrow \ldots$$

such that $\partial_{n-1}' \partial_n' = 0$ and $\partial_{n-1}\partial_n = 0$.

Since $\varepsilon' h_0 - \varepsilon' t\varepsilon = \gamma\varepsilon - \gamma\varepsilon = 0$ we have

$$\varepsilon'(h_0 - t\varepsilon) = 0. \tag{5.5}$$

This implies that $\mathrm{Im}(h_0 - t\varepsilon) \subset \mathrm{Kern}\ \varepsilon' = \mathrm{Im}\partial'_1$. Hence, since X_0 is projective, from the following diagram

$$\begin{tikzcd} & X_0 \arrow[dl, dashed, "s_0"'] \arrow[d, "h_0 - t\varepsilon"] & \\ X'_1 \arrow[r, "\partial'_1"] & X'_0 \arrow[r, "\varepsilon'"] & C' \end{tikzcd}$$

we have an homomorphism $s_0 : X_0 \to X'_1$ such that: $\partial'_1 s_0 = h_0 - t_\varepsilon$.

Having constructed s_0 we proceed by induction. We want to find

$$s_n : X_n \to X'_{n+1} \text{ s.t. } \partial'_{n+1} s_n = h_n - s_{n-1}\partial_n.$$

We have

$$\partial'_n(h_n - s_{n-1}\partial_n) = \partial'_n h_n - \partial'_n s_{n-1}\partial_n = h_{n-1}\partial_n - (h_{n-1} - s_{n-2}\partial_{n-1})\partial_n =$$
$$= h_{n-1}\partial_n - h_{n-1} + s_{n-2}\partial_{n-1}\partial_n = 0$$

since, by the induction hypothesis $\partial'_n s_{n-1} = h_{n-1} - s_{n-2}\partial_{n-1}$ and $\partial\partial = 0$.

Thus $\mathrm{Im}(h_n - s_{n-1}\partial_n) \subset \mathrm{Kern}\ \partial'_n = \mathrm{Im}\ \partial'_{n+1}$. Hence from the diagram

$$\begin{tikzcd} & X_n \arrow[dl, dashed, "s_n"'] \arrow[d, "h_n - s_{n-1}\partial_n"] & \\ X'_{n+1} \arrow[r, "\partial'_{n+1}"] & X'_n \arrow[r, "\partial'_n"] & X'_{n-1} \end{tikzcd}$$

we construct s_n, using the projectivity of X_n. $\square$

THEOREM 5.1 - <u>Suppose that</u> $X, X', C, C', \varepsilon, \varepsilon', \gamma$ <u>are defined as in the previous Lemma</u>.

Then there exists a chain transformation $f\colon X \to X'$ with $\varepsilon' f_0 = \gamma\varepsilon$ and any two such chain transformations are chain homotopic.(11)

Proof. Since X_0 is projective and ε' is an epimorphism we can find $f_0\colon X_0 \to X_0'$ with $\varepsilon' f_0 = \gamma\varepsilon$.

Using the same induction argument of the previous Lemma we can construct $f_n\colon X_n \to X_n'$ such that $\partial_n' f_n = f_{n-1}\partial_n$.

Now suppose that f and g are two chain transformations with the property:

$$\varepsilon' f_0 = \gamma\varepsilon \quad \text{and} \quad \varepsilon' g_0 = \gamma\varepsilon$$

Then: $\varepsilon'(f - g) = 0\varepsilon = 0$.

Hence, applying the previous Lemma with $f - g = h$, $\gamma = 0$, $t = 0$ we obtain the existence of homomorphisms $s_n\colon X_n \to X_{n+1}'$ such that:

$$\partial_{n+1}' s_n + s_{n-1}\partial_n = f_n - g_n$$

that is f and g are chain homotopic. □

Theorem 5.1, as well as Lemma 5.1, can be proved under a little more general hypotheses, see [11].

THEOREM 5.2 - If $(X \xrightarrow{\xi} C)$ and $(X' \xrightarrow{\xi'} C)$ are two projective resolutions of C, and A is any R-module, then:

$$H^n(X,A) \cong H^n(X',A)$$

Proof. Consider the identity $1_C\colon C \to C$.

---

(11) Two chain transformations $f,g\colon X \to X'$ are chain homotopic if there exists a family of module homomorphisms: $s_n\colon X_n \to X_{n+1}'$ such that:

$$\partial_{n+1}' s_n + s_{n-1}\partial_n = f_n - g_n.$$

From Theorem 5.1 we have two chain transformations: $f\colon\ X \to X'$ and $g\colon\ X' \to X$ such that:

$$\varepsilon' f_0 = \varepsilon \quad \text{and} \quad \varepsilon g_0 = \varepsilon'.$$

Hence $gf\colon\ X \to X$ and $fg\colon\ X' \to X'$ have the properties: $\varepsilon(gf) = \varepsilon$ and $\varepsilon'(fg) = \varepsilon'$.

Consequently, by the previous theorem, they are chain homotopic to the identities $1_X\colon\ X \to X$ and $1_{X'}\colon\ X' \to X'$, respectively.

Considering the induced homomorphisms:

$$f^*\colon\ H^n(X',A) \to H^n(X,A) \quad \text{and} \quad g^*\colon\ H^n(X,A) \to H^n(X',A).$$

we have $g^*f^* = 1_{H^n(X',A)}$ and $f^*g^* = 1_{H^n(X,A)}$ because gf and fg are chain homotopic to the identities (see [11] or [16]).

Hence f^* (or g^*) is an isomorphism. □

REMARK 5.1 - Since:

$$X_1 \to X_0 \xrightarrow{\varepsilon} C \to 0$$

is right exact, then:

$$0 \to \operatorname{Hom}(C,A) \xrightarrow{\varepsilon^*} \operatorname{Hom}(X_0,A) \to \operatorname{Hom}(X_1,A)$$

is left exact. This proves that $H^0(X,A) \cong \operatorname{Hom}(C,A)$.

A resolution <u>under</u> the R-module A is an exact sequence of R-modules:

$$(5.6) \qquad 0 \to A \xrightarrow{\varepsilon} Y^0 \xrightarrow{\delta_0} Y^1 \xrightarrow{\delta_1} Y^2 \ \dots \to Y^n \xrightarrow{\delta_n} Y^{n+1} \to \dots.$$

The resolution is called injective if every Y^n is injective.(12)

Since every R-module A is a submodule of an injective R-module, there exists at least one injective resolution of A.

---

(12) An R-module J is injective if given a monomorphism $i\colon\ A \to B$ and an homomorphism $\alpha\colon\ A \to J$ there exists an homomorphism $\beta\colon B \to J$ such that the following diagram in commutative:

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "\alpha"'] & B \arrow[dl, dashed, "\beta"] \\ J & \end{tikzcd}$$

Fixing an R-module C we can apply the covariant functor $\mathrm{Hom}_R(C,-)$ to (5.6) obtaining:

(5.7) $$0 \to \mathrm{Hom}_R(C,A) \xrightarrow{\varepsilon^*} \mathrm{Hom}_R(C,Y^0) \to \mathrm{Hom}_R(C,Y^1) \to \cdots$$

which is, in general, not exact.

The measure of the nonexactness of (5.7) gives the cohomology: $H^n(C,Y) = H^n(\mathrm{Hom}_R(C,Y))$.

As for the projective resolutions it is possible to prove that $H^n(C,Y)$ depends only on C and A and not on the particular injective resolution under A.

Moreover (see [11]):

THEOREM 5.3 - <u>Suppose that A and C are two R-modules. For any projective resolution</u> $(X \xrightarrow{\xi} C)$ <u>and for any injective resolution</u> $(A \xrightarrow{\eta} Y)$ <u>we have:</u>

(5.8) $$H^n(X,A) \cong H^n(C,Y) \quad n = 0,1,\ldots.$$

The group $H^n(X,A)$(or $H^n(C,Y)$) is also called the n-th extension group of A by C and denoted by $\mathrm{Ext}^n(C,A)$.

From now on let G be a group, written multiplicatively. The free abelian group $\mathbb{Z}[G]$ generated by the elements $g \in G$, is the set of the finite sums:

$$\sum_g m(g)g \quad g \in G, \quad m(g)\in\mathbb{Z}$$

Thus an element in $\mathbb{Z}[G]$ is a function $m\colon G \to \mathbb{Z}$ which is zero except for a finite number of $g \in G$.

It is possible to define also a product:

$$\left(\sum_g m(g)g\right)\left(\sum_\gamma m'(\gamma)\gamma\right) = \sum_{g\gamma} m(g)m'(\gamma)g\gamma \quad g,\gamma \in G$$

so that $\mathbb{Z}[G]$ becomes a ring called the (integral) group ring of G.

A ring homomorphism $\varepsilon\colon \mathbb{Z}[G] \to \mathbb{Z}$, called an augmentation, is defined by setting:

$$\varepsilon\left(\sum_g m(g)g\right) = \sum_g m(g)$$

Modules over $\mathbb{Z}[G]$ will be called G-modules.

If $G = C_m(t)$, the multiplicative cyclic group of order m with generator t, then

the group ring $\Gamma = \mathbb{Z}[C_m(t)]$ is the ring of all polynomials $u = \sum_{i=0}^{m-1} a_i t^i$ in t, with integral coefficients a_i, taken modulo the relation $t^m = 1$.

If G is the infinite cyclic group with generator t, then $\mathbb{Z}(G)$ is the ring of polynomials $\sum_i a_i t^i$, $i \in \mathbb{Z}$ and only finite $a_i \neq 0$.

An abelian group A is given a unique structure as a (left) G-module by giving either:

(i) a function θ: $G \times A \to A$, $\theta(g,a) = ga$, $g \in G$, $a \in A$ such that:

$$\begin{cases} g(a_1 + a_2) = ga_1 + ga_2 \\ (g_1 g_2)a = g_1(g_2 a) \\ 1a = a \end{cases}$$

(ii) a group homomorphism

$$\phi\colon\ G \to \mathrm{Aut}\ A.$$

The definition of a G-module A, essentially means that there is an action of G on A which also considers the algebraic structure of A.

In particular any abelian group A can be regarded as a trivial G-module, considering the trivial action of G on A: $ga = a$, $\forall g \in G$.

Now let G^n be the cartesian product of n copies of G. We denote by P_n the free abelian group on G^{n+1} made into a G-module by the "action":

$$g(g_0, g_1, \ldots, g_n) = (gg_0, gg_1, \ldots, gg_n).$$

We can define maps ∂_n: $P_n \to P_{n-1}$, $n = 1,2,\ldots$, by:

$$(g_0, \ldots, g_n) \dashrightarrow \sum_{i=0}^{n} (-1)^i \ (g_0, \ldots, \hat{g}_i, \ldots, g_n)$$

where $\wedge$ indicates deletion.

In particular $P_0 = \mathbb{Z}[G]$.

THEOREM 5.4. <u>If ε: $P_0 \to \mathbb{Z}$ is the augmentation then the sequence:</u>

$$P = \ldots \to P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

<u>is a free resolution of $\mathbb{Z}$, where $\mathbb{Z}$ is a trivial G-module.</u>

<u>Proof</u>. First of all we construct the functions:

$s_{-1}: \mathbb{Z} \to P_0$, $s_{-1}(1) = \bar{1}$, $1 \in \mathbb{Z}$, $\bar{1}$ = identity of G,

$s_n : P_n \to P_{n-1}$, $s_n(g_0,\dots,g_n) = (\bar{1},g_0,\dots,g_n)$

which are group-homomorphisms.

Then from the definition and some easy computations we have:

(5.9)
$$\begin{cases} \varepsilon s_{-1} = 1_{\mathbb{Z}} \\ \partial_{n+1}s_n + s_{n-1}\partial_n = 1_{P_n} \quad n \geq 0 \end{cases}$$

(5.9) implies that the chain maps 1 and 0: $\mathbb{P} \to \mathbb{P}$ are chain homotopic.

Hence the induced map $1^*, 0^*$: $H_n(\mathbb{P}) \to H_n(\mathbb{P})$ between the homology groups of the chain complex $\mathbb{P}$ are equal, that is $\mathbb{P}$ has a trivial homology and consequently $\mathbb{P}$ is a resolution of abelian groups over $\mathbb{Z}$.

$\mathbb{P}$ is also a resolution of G-modules over $\mathbb{Z}$ because if $\mathbb{P}$ is exact as sequence of abelian groups then since ∂_n are module homomorphisms it is exact also as sequence of G-modules.

Moreover $\mathbb{P}$ is a free resolution by the construction of P_n. $\square$

We remark that P_n is isomorphic (as a G-module) to the tensor product (over $\mathbb{Z}$) of $\mathbb{Z}G$ with itself $n + 1$ times.

There is another way of constructing a free resolution of $\mathbb{Z}$, which is useful in the applications.

We can define Q_n $(n > 0)$ as the free G-module with generators $[g_1,\dots,g_n]$, all n-tuples of elements of G, and Q_0 as the free G-module on the single generator $[\ ]$.

For each $n \geq 0$ we define the functions:

$$\tau: P_n \to Q_n \quad \text{and} \quad \sigma: Q_n \to P_n$$

$$\tau(g_0,\dots,g_n) = g_0[g_0^{-1}g_1, g_1^{-1}g_2,\dots,g_{n-1}^{-1}g_n]$$

$$\sigma[g_1,\dots,g_n] = (1,g_1,g_1g_2,g_1g_2g_3,\dots,g_1g_2\cdots g_n)$$

They are inverse to one another, hence P_n and Q_n are isomorphic.

Thus the following commutative diagram defines univocally the maps

$d_n: Q_n \to Q_{n-1}, \quad n \geq 1$

$$\begin{array}{ccc} P_n & \xrightarrow{\tau} & Q_n \\ {\scriptstyle \partial_n}\downarrow & & \downarrow{\scriptstyle d_n} \\ P_{n-1} & \xrightarrow{\tau} & Q_{n-1} \end{array}$$

$$d_n[g_1,\ldots,g_n] = g_1[g_2,\ldots,g_n] + \sum_{i=1}^{n-1} (-1)^i [g_1,\ldots,g_i g_{i+1},\ldots,g_n] +$$

$$+ (-1)^n [g_1,\ldots,g_{n-1}]$$

In particular:

$$d_1[g] = g[\ ] - [\ ], \quad d_2[g_1,g_2] = g_1[g_2] - [g_1g_2] + [g_1].$$

Since $\tau: \mathbb{P} \to \mathbb{Q}$ induces an isomorphism between the homology groups $H_n(\mathbb{P}) \to H_n(\mathbb{Q})$, Theorem 5.4 implies the following:

THEOREM 5.5 - $\mathbb{Q} = \ldots \to Q_2 \xrightarrow{d_2} Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ <u>is a free resolution of</u> $\mathbb{Z}$.

DEFINITION 5.1 - <u>Given any G-module A, we define the cohomology groups</u> $H^n(G,A)$ <u>of G with coefficients in A as:</u>

$$H^n(G,A) = H^n(\mathbb{P},A) \cong H^n(\mathbb{Q},A)$$

We remember that $H^n(\mathbb{P},A) = H^n(\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{P},A))$, and the *analogous definition holds for* $H^n(\mathbb{Q},A)$ that is the dependence on the group G, in the definition 5.1, enters in the structure of A and P_n or Q_n, as G-modules.

Moreover, from Theorem 5.3 we have that $H^n(G,A)$ depends only on G, and A, since $\mathbb{Z}$ is fixed, and can be computed from any projective resolution of G-modules over $\mathbb{Z}$ or any injective resolution of G-modules under A.

Since Q_n is a free G-module with generators $[g_1,\ldots,g_n]$, an element $f: Q_n \to A$

in $\mathrm{Hom}_{\mathbb{Z}[G]}(Q_n,A)$ is a G-module homomorphism which is uniquely determined by its values on these generators.

Therefore $\mathrm{Hom}_{\mathbb{Z}[G]}(Q_n,A)$ can be identified with the set of all those functions f (n-cochains) on n-arguments with values in A.

The addition of two cochains is given by:

$$(f_1 + f_2)(g_1,\dots,g_n) = f_1(g_1,\dots,g_n) + f_2(g_1,\dots,g_n).$$

The coboundary homomorphisms $\delta^n: Q_n \to Q_{n+1}$ are defined by:

$$\delta^n f(g_1,\dots,g_{n+1}) = (-1)^{n+1}[g_1 f(g_2,\dots,g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_1,\dots,g_i g_{i+1},\dots,g_{n+1}) + (-1)^{n+1} f(g_1,\dots,g_n)].$$

Then $H^n(G,A)$ is the n-th cohomology group of the complex $\mathrm{Hom}_{\mathbb{Z}[G]}(Q_n,A)$ with this coboundary map.

THEOREM 5.6 - If G is a finite group of order k, every element of $H^n(G,A)$, $n > 0$, has order dividing k.

Proof. For each n-cochain f we define an (n-1)-cochain

$$h(g_1,\dots,g_{n-1}) = \sum_{g\in G} f(g_1,\dots,g_{n-1},g)$$

The theorem is proved if we show that, for $f \in H^n(G,A)$, $kf = 0$ that is $kf \in \mathrm{Im}\delta^{n-1}$.

We have:

$$\sum_{g\in G} \delta^n f(g_1,\dots,g_n,g) = -\delta^{n-1} h(g_1,\dots,g_n) + kf(g_1,\dots,g_n) = 0,$$

since $\delta^n f = 0$, that is $kf = \delta^{n-1} h \in \mathrm{Im}\delta^{n-1}$. □

COROLLARY 5.1 - Let G be a finite group and D a divisible[13] abelian group with

(13) A group G is said to be divisible if the equation $mx = g$ has a solution $x \in G$, for every given $g \in G$ and $m \in \mathbb{Z}$.

no elements of finite order. If D has any structure of G-module, then $H^n(G,D) = 0$, for $n > 0$.

Proof. We take f and h as in the previous proof.

Since D is divisible, it is possible to find an $(n-1)$ cochain q such that: $h = kq$.

Then we have: $kf = \delta h = \delta kq = k\delta q$.

But if $k(f-\delta q) = 0$, then $f = \delta q$ because D does not have any element of finite order.

This means that every cocycle f is a coboundary and hence $H^n(G,D) = 0$. $\square$

6. Equivariant cohomology in presence of finite groups.

The knowledge of the classifying space BG of a group G is one of the main steps in computing the equivariant cohomology of a G-invariant manifold M, using the fibration (4.5).

In this section we give an interpretation of the cohomology of BG, when G is a finite group, which allows us to compute $H^*(BG)$, for any finite abelian group.

Let us suppose that X is a topological space and a group G acts properly on it. In this case we can consider a base in X, made up of proper sets U (see section 1).

Using the projection $p: X \to X/G$, $p(x) = O(x)$, these sets determine, for the topology on X/G, the open sets $pU = V$ which will be called proper sets in X/G.

Then X is a covering space for X/G, under the projection p.

In fact, by definition of proper action, each $p^{-1}V$, $V = pU \subset X/G$, is the union of disjoint sets gU, $g \in G$ and the restriction of p to each gU is an homeomorphism between gU and $p(gU)$.

Now we consider the singular homology of X with $\mathbb{Z}$ coefficients. We denote by $S(X) = \{S_n(X)\}$ the complex of abelian groups $S_n(X)$ generated by the singular n-simplices $T: \Delta_n \to X$[14], with the usual boundary homomorphisms.

We have:

THEOREM 6.1 - If G acts properly on X, S(X) is a complex of free G-modules.

Proof. Giving the "action":

$$G \times S_n(X) \to S_n(X), \quad (g,T) \to gT \in S_n(X)$$

we make $S_n(X)$ a G-module.

Moreover it is easy to see that the boundary homomorphisms are G-module homomorphisms.

---

(14) Δ_n is the standard affine n-simplex in $\mathbb{R}^n$.

Finally if $X_0 \subset X$ is a subset containing exactly one point from each orbit, the set of singular n-simplices T with initial vertex in X_0 is a set of free generators for $S_n(X)$ as a G-module. □

THEOREM 6.2 - If G acts properly on X, any n-simplex $\bar{T}$ in X/G is the image, under p, of some T in X. Moreover these T's can be taken in a set of free generators of $S_n(X)$ as a G-module.

Proof: The n-simplex $\bar{T}$ is a map from $\Delta_n \to X/G$. By the "lifting" property(15) in a covering space, $\bar{T}$ can be lifted to a map $T\colon \Delta_n \to X$ such that $pT = \bar{T}$. □

Now we suppose that A is an abelian group with the trivial structure of a G-module.

THEOREM 6.3 - If G acts properly on X then:

$$\mathrm{Hom}_{\mathbb{Z}}(S(X/G),A) \cong \mathrm{Hom}_{\mathbb{Z}[G]}(S(X),A)$$

and hence:

$$H^n(X/G,A) \cong H^n(\mathrm{Hom}_{\mathbb{Z}[G]}(S(X),A))$$

Proof. The induced map:

$$p^*\colon\ \mathrm{Hom}_{\mathbb{Z}}(S(X/G),A) \to \mathrm{Hom}_{\mathbb{Z}[G]}(S(X),A) \qquad (p^*f)\,T = f(pT)$$

is an isomorphism.

In fact any cochain $f \in \mathrm{Hom}_{\mathbb{Z}}(S_n(X/G),A)$ is uniquely determined by assigning its values on $\bar{T}$, an n-simplex in $S_n(X/G)$, while a cochain f' in $\mathrm{Hom}_{\mathbb{Z}[G]}(S_n(X),A)$ is

(15) The "lifting" property asserts that if $p\colon\ X \to B$ is a covering space and f is a map from $Y \to B$ then there exists a map f', $f'\colon\ Y \to X$ such that the following diagram is commutative:

$$\begin{tikzcd} & X \arrow[d, "p"] \\ Y \arrow[ur, dashed, "f'"] \arrow[r, "f"'] & B \end{tikzcd}$$

This proposition is a particular case of a general property for fibrations. (see [16])

uniquely determined by its values on a set of generators $T \in S_n(X)$. By the previous Lemma these generators are in 1-1 correspondence. Therefore the assertion follows. $\square$

Before stating the last Theorem we recall that a space X is acyclic if it has the homology of a point, that is: $P_t(X) = 1$.

THEOREM 6.4 - <u>If G acts properly on an acyclic space X, then:</u>

$$H^n(X/G,A) \cong H^n(G,A) \quad n \geq 0 \tag{6.1}$$

<u>Proof</u>. Since X is acyclic the sequence of G-modules:

$$\cdots\cdot \to S_2(X) \to S_1(X) \to S_0(X) \to \mathbb{Z} \to 0$$

is a free resolution of $\mathbb{Z}$.

Then, by the definition of cohomology of a group,

$$H^n(\mathrm{Hom}_{\mathbb{Z}[G]}(S(X),A)) \cong H^n(G,A).$$

The previous theorem gives (6.1). $\square$

We will apply this result to compute the cohomology of the classifying space of a finite abelian group.

Let us suppose that G is a finite group. We know that there is an universal G-bundle with a contractible total space E on which G acts freely, and hence properly, since G is finite. Then the cohomology of the classifying space of G, $BG = E/G$ can be obtained from (6.1) computing the cohomology of G with coefficients in a trivial G-module A.

We suppose at first that G is the multiplicative cyclic group $C_m(t)$, of order m, with generator t.

We have already observed that $\Gamma = \mathbb{Z}[C_m(t)]$ is the ring of polynomials

$$u = \sum_{i=0}^{m-1} a_i t^i, \quad a_i \in \mathbb{Z}, \quad \text{modulo the relation } t^m = 1.$$

Two particular elements in Γ are:

$$N = 1 + t + \ldots + t^{m-1} \quad \text{and} \quad D = t - 1$$

They have the properties:

i) $ND = 0$

ii) $Nu = 0 \iff u = Dv$ for some $v \in \Gamma$

iii) $Du = 0 \iff u = Na_0$.

From i), ii), iii) it follows that the sequence:

$$\xrightarrow{D_*} \Gamma \xrightarrow{N_*} \Gamma \xrightarrow{D_*} \Gamma \xrightarrow{\varepsilon} Z \to 0$$

is a free resolution of $\mathbb{Z}$, defining $D_*u = Du$, $N_*u = Nu$, $\varepsilon(u) = \sum a_i$.

Let A be any G-module. The group $\mathrm{Hom}_\Gamma(\Gamma,A)$ is isomorphic to A by the isomorphism α: $\mathrm{Hom}_\Gamma(\Gamma,A) \to A$, $\alpha(f) = f(1)$.

The corresponding sequence obtained on applying $\mathrm{Hom}_\Gamma(-,A)$ is:

$$0 \to \mathrm{Hom}(\mathbb{Z},A) \xrightarrow{\varepsilon^*} A \xrightarrow{-D^*} A \xrightarrow{N^*} A \xrightarrow{-D^*} A \to$$

$D^*a = Da = (t - 1)a \qquad a \in A$

$N^*a = Na = (1 + t + \ldots + t^{m-1})a \qquad a \in A.$

Then Kern $D^* = [a \mid ta = a]$ and Kern $N^* = [a \mid a + ta + \ldots + t^{m-1}a = 0]$.

Hence we have the following Theorem:

THEOREM 6.5 - <u>The cohomology groups of</u> $C_m(t)$ <u>with coefficients in</u> A <u>are:</u>

$$H^0(C_m(t),A) = [a \mid ta = a] \cong \mathrm{Hom}_\Gamma(\mathbb{Z},A)$$

$$H^{2n}(C_m(t),A) = [a \mid ta = a]/N^*A$$

$$H^{2n+1}(C_m(t),A) = [a \mid Na = 0]/D^*A$$

In particular if A is a trivial G-module from Theorem 6.5 we have:

$$H^0(C_m(t),A) = A \cong H^0(BC_m(t),A)$$

$$H^{2n}(C_m(t),A) = A/mA \cong H^{2n}(BC_m(t),A)$$

$$H^{2n+1}(C_m(t),A) = [a \mid ma = 0] \cong H^{2n+1}(BC_m(t),A)$$

since $\{a \mid ta = a\} = A$, $\{a \mid Na = 0\} = \{a \mid ma = 0\}$, $N^*A = \{Na \mid a\in A\} = \{ma\}_{a\in A}$ and $D^*A = \{(t-1)a \mid a \in A\} = 0$

For example if $C_m(t) = \mathbb{Z}_p$ and A is $\mathbb{Z}_p$ considered as a trivial $\mathbb{Z}_p$-module the Poincaré series, expressing the cohomology of $\mathbb{Z}_p$ with coefficients in $\mathbb{Z}_p$ is:
$P_t(\mathbb{Z}_p) = 1 + t + t^2 + \ldots = \frac{1}{1-t} = H^*(B\mathbb{Z}_p,\mathbb{Z}_p)$.

Now, let us suppose that G is a finite abelian group. Then G can be decomposed into a direct sum of cyclic groups G_i of order z_i:

$$G = G_1 \oplus \ldots \oplus G_n$$

with z_i dividing z_{i+1} and this decomposition is unique up to isomorphism.
Then we have

$$H^*(G_1,\mathbb{Z}_p) \otimes \ldots \otimes H^*(G_n,\mathbb{Z}_p) = \tag{6.2}$$
$$= H^*(G,\mathbb{Z}_p) = H^*(BG,\mathbb{Z}_p)$$

for any p dividing z_n, where $\mathbb{Z}_p$ is considered as a trivial G-module.

The result (6.2) can be expressed in a more general form. We refer to [9], Chapter III, for more details.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER #2717	2. GOVT ACCESSION NO. AD-A144661
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle) MORSE THEORY FOR FLOWS IN PRESENCE OF A SYMMETRY GROUP	5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Filomena Pacella	8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709	12. REPORT DATE July 1984
	13. NUMBER OF PAGES 48
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report) UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Morse theory
group actions
equivariant flows
equivariant cohomology

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper contains results reported in a series of seminars given by the author at the University of Wisconsin-Madison. These concern Morse theory in the presence of symmetry. Different ways of studying an equivariant flow are investigated and, in particular, the equivariant Morse theory for flows is described.

This theory requires results on the cohomology of classifying spaces for finite groups which are also described here.

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